Homework 7a
Algebra

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Note on notation: For a lot of this assignment, I've deliberality omitted composition symbols ($\circ$) for function compositions in order to reduce visual clutter.

Lemma 0.1 (for Exercise 1)**.** *Let A be a ring and let M be an A-module. Then* $\mathrm{End}_A(M)$ *is a ring.*

Proof. We assume here that M is a left A-module, but the same arguments work for a right A-module. We first recall the definition:

$$\mathrm{End}_A(M) = \{f : M \to M | f \text{ is an } A\text{-module homomorphism}\}$$

where addition and multiplication are defined by

$$(f+g)(x) = f(x) + g(x) \qquad (fg)(x) = (f \circ g)(x)$$

First we check that $\mathrm{End}_A(M)$ is a ring. We should check that the sum and composition of A-module homomorphisms is an A-module homomorphism, which simply means checking that $f+g$ and $f \circ g$ preserve addition in M and multiplication from A. Let $f, g \in \mathrm{End}_A(M)$ and $x, y \in M$ and $a \in A$. Preservation of addition for $f+g$:

$$\begin{aligned}(f+g)(x+y) &= f(x+y) + g(x+y) \\ &= f(x) + f(y) + g(x) + g(y) \\ &= (f+g)(x) + (f+g)(y)\end{aligned}$$

Preservation of scalar multiplication from A for $f+g$:

$$(f+g)(ax) = f(ax) + g(ax) = af(x) + ag(x) = a(f(x) + g(x)) = a(f+g)(x)$$

Preservation of addition for $f \circ g$:

$$\begin{aligned}(f \circ g)(x+y) &= f(g(x+y)) = f(g(x) + g(y)) \\ &= f(g(x)) + f(g(y)) = (f \circ g)(x) + (f \circ g)(y)\end{aligned}$$

Preservation of scalar multiplication from A for $f \circ g$:

$$(f \circ g)(ax) = f(g(ax)) = f(ag(x)) = af(g(x)) = a(f \circ g)(x)$$

Thus $\mathrm{End}_A(M)$ is closed under the defined multiplication and addition operations. Now we show that it is abelian group under addition. The zero map $x \mapsto 0$ is the additive identity, and the additive inverse of $x \mapsto f(x)$ is the map $x \mapsto -f(x)$. Associativity and commutativity follow immediately from those properties for addition in M, for example commutativity:

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x) \implies f+g = g+f$$

Thus $\mathrm{End}_A(M)$ is an abelian group under addition. The multiplication is associative because function composition is always associative, and the multiplicative identity is the identity map on M, $x \mapsto x$. Finally, addition distributes over addition because

$$(f \circ (g+h))(x) = f(g(x) + h(x)) = (f \circ g)(x) + (f \circ h)(x)$$

using the ring homomorphism properties of f. □

Proposition 0.2 (Exercise 1)**.** *Let A be a ring and let M be a simple A-module. Then $\mathrm{End}_A(M)$ is a division ring.*

Proof. We showed above that $\mathrm{End}_A(M)$ is a ring, so we just need to show that every $f \in \mathrm{End}_A(M)$ (except the zero map) has a multiplicative inverse, that is, we need to show that f is invertible. Then since a bijective A-module homomorphism is an isomorphism, $f^{-1} \in \mathrm{End}_A(M)$ and we have the required inverse.

Let $f \in \mathrm{End}_A(M)$ be nonzero. Note that since f is an A-module endomorphism, the image and kernel are submodules of M. Since f is nonzero, the kernel is not all of M, so the kernel is a proper submodule. Since M is simple, the only proper submodule is $\{0\}$, so f has trivial kernel, so f is injective. Similarly, since f is not the zero map, the image is a nontrivial submodule, but the only nontrivial submodule is all of M. Hence f is surjective.

Thus f is bijective, so it is invertible. As already noted, this means that $f^{-1} \in \mathrm{End}_A(M)$, so f has a multiplicative inverse. Hence $\mathrm{End}_A(M)$ is a division ring. □

Proposition 0.3 (Exercise 3)**.** *Let R be an integral domain containing a field k as a subring, and suppose that R is a finite dimensional vector space over k under the ring multiplication. Then R is a field.*

Proof. Let a be a nonzero element of R and define $f_a : R \to R$ by $x \mapsto ax$. Then f_a is a k-linear map, because for $x, y \in R$ and $\lambda \in k$ we have

$$\begin{aligned} f_a(x+y) &= a(x+y) = ax + ay = f_a(x) + f_a(y) \\ f_a(\lambda x) &= a\lambda x = \lambda a x = \lambda f_a(x) \end{aligned}$$

The kernel of f_a is trivial because R has no nontrivial zero divisors:

$$\ker f = \{x \in R : ax = 0\} = \{0\}$$

Thus $\dim \ker f = 0$, so by the Rank-Nullity Theorem, $\dim \operatorname{im} f = \dim R$. Thus $\operatorname{im} f$ is a subspace of R of equal dimension. Since R is finite-dimensional, this implies that f is surjective. In particular, there exists $x \in R$ so that $f(x) = ax = 1$. Hence a is a unit, so R is a field. □

Proposition 0.4 (Exercise 4a, part one)**.** *Let R be a ring and let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules. Then the following are equivalent:

1. *There exists a homomorphism $\psi : M \to M'$ such that $\psi \circ f = \mathrm{id}_{M'}$.*
2. *There exists a homomorphism $\phi : M'' \to M$ such that $g \circ \phi = \mathrm{id}_{M''}$.*
3. *There is an isomorphism $\theta : M \to M' \oplus M''$ such that the following diagram commutes: (ι is the inclusion $x \mapsto (x, 0)$ and π is the projection $(x, y) \mapsto y$.)*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\mathrm{id}_{M'}"'] & M \arrow[r, "g"] \arrow[d, "\theta"'] & M'' \arrow[r] \arrow[d, "\mathrm{id}_{M''}"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "\iota"] & M' \oplus M'' \arrow[r, "\pi"] & M'' \arrow[r] & 0
\end{tikzcd}$$

Proof. First we show (1) $\implies$ (3). Let $\psi : M \to M'$ be the given homomorphism. Define $\theta : M \to M' \oplus M''$ by $m \mapsto (\psi m, gm)$. Then θ is an R-module homomorphism, and the diagram commutes since

$$\begin{aligned}
\theta \circ f(m') &= (\psi f(m'), gf(m')) = (m', 0) = \iota \circ \mathrm{id}_{M'}(m') \\
\pi \circ \theta(m) &= g(m) = \mathrm{id}_{M''} \circ g(m)
\end{aligned}$$

using the fact that $gf = 0$ by exactness. If $m \in \ker \theta$, then $\psi m = 0$ and $gm = 0$. Then by exactness, $m \in \operatorname{im} f$ so there exists $m' \in M'$ with $f(m') = m$. Then $0 = \psi m = \psi \circ f(m') = m'$ so $m = f(m') = f(0) = 0$. Hence θ has trivial kernel, so it is injective.

Now we show θ is surjective. Let $(m', m'') \in M' \oplus M''$. Since g is surjective, there exists $m \in M$ so that $g(m) = m''$. Then since $g \circ f = 0$, we have $g(m + f(x')) = g(m) = m''$ for all $x' \in M'$. Then define $x' = m' - \psi(m)$, so

$$m' = \psi(m) + x' = \psi(m) + \psi(f(x')) = \psi(m + f(x'))$$

Then

$$\theta(m + f(x')) = (\psi(m + f(x')), g(m + f(x'))) = (m', m'')$$

Thus θ is surjective, so it is an isomorphism making the diagram commute. This completes the proof of (1) $\implies$ (3).

Now we show (3) $\implies$ (1). Let $\theta : M \to M' \oplus M''$ given by $m \mapsto (\theta_1(m), \theta_2(m))$ be an isomorphism making the diagram commute. By commutativity of the right square, $\theta_2(m) = g(m)$. Let $\psi(m) = \theta_1(m)$. Then ψ is an R-module homomorphism, and by commutativity of the left square

$$(\psi \circ f(m'), g \circ f(m')) = \theta \circ f(m') = \iota \circ \mathrm{id}_{M'}(m') = (m', 0)$$

Thus $\psi \circ f = \mathrm{id}_{M'}$. This completes (3) $\implies$ (1).

Now we show (2) $\implies$ (3). Let ϕ be the given homomorphism. First we construct an isomorphism going in the opposite direction. Define $h : M' \oplus M'' \to M$ by $(m', m'') \mapsto f(m') + \phi(m'')$. Then h inherits the needed module homomorphism properties from f and ϕ. If $(m', m'') \in \ker h$, then since $gf = 0$ and $g\phi = \mathrm{id}_{M'}$ we have

$$0 = f(m') + \phi(m'') \implies 0 = g(f(m') + \phi(m'')) = gf(m') + g\phi(m'') = m''$$

so $m'' = 0$. Then $0 = f(m')$, which implies $m' = 0$ since f is injective. Thus h has trivial kernel. Now we show that h is surjective. Let $m \in M$, and define $m'' = g(m)$. Now consider

$$g(m - \phi gm) = gm - g\phi gm = gm - gm = 0$$

so $m - \phi gm \in \ker g = \operatorname{im} f$. Thus there exists $m' \in M'$ so that $f(m') = m - \phi gm$. Thus

$$h(m', m'') = f(m') + \phi(m'') = (m - \phi gm) + \phi gm = m$$

Thus h is surjective, so it is an isomorphism. Now define $\theta = h^{-1}$. By the bijectivity of h, every $m \in M$ can be written uniquely as $m = f(m') + \phi(m'')$, so after decomposing

$$\theta(m) = \theta(f(m') + \phi(m'')) = (m', m'')$$

We show that the left square commutes. Let $m' \in M'$. Then $f(m') = f(m') + \phi(0)$ so

$$\theta \circ f(m') = \theta(f(m') + \phi(0)) = (m', 0) = \iota \circ \mathrm{id}_{M'}(m')$$

Now we show that the right square commutes. Let $m \in M$ and write it as $m = f(m') + \phi(m'')$. Then

$$\mathrm{id}_{M''} \circ g(m) = g(f(m') + \phi(m'')) = gf(m') + g\phi(m'') = m'' = \pi \circ \theta(m)$$

using the fact that $gf = 0$ and $g\phi = \mathrm{id}_{M''}$. Thus θ is the required isomorphism. This completes (2) $\implies$ (3).

Finally, we show (3) $\implies$ (2). Let θ be the given isomorphism. Let $\chi : M'' \to M' \oplus M''$ be the inclusion $m'' \mapsto (0, m'')$. Then define $\phi : M'' \to M$ by $\phi = \theta^{-1}\chi$. Then ϕ is an R-module homomorphism. By commutativity of the diagram, $\pi\theta = g$ so we have

$$g\phi = \pi\theta\theta^{-1}\chi = \pi\chi = \mathrm{id}_{M''}$$

This completes the proof of the equivalence of (1), (2), and (3). $\square$

Proposition 0.5 (Exercise 4a, part two)**.** *Let R be a ring and let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules. Then the following are equivalent:

1. *The sequence splits (there is a homomorphism $\phi : M'' \to M'$ so that $g \circ \phi = \mathrm{id}_{M''}$).*
2. *There exists a submodule N of M such that $M = \operatorname{im} f \oplus N$.*

Furthermore, if such a submodule N exists, then $N \cong M''$.

Proof. First we prove (1) $\implies$ (2). Let $\phi : M'' \to M'$ be the given R-module homomorphism. As shown during that proof (in showing (2) $\implies$ (3)), every $m \in M$ can be written uniquely as $m = f(m') + \phi(m'')$ where $m' \in M'$ and $m'' \in M''$. Thus $M = \operatorname{im} f \oplus \phi(M'')$, so we can take $N = \phi(M'')$. (This is a submodule because ϕ is a homomorphism.)

Now we show (2) $\implies$ (1). Let N be the given submodule. For $m \in M$ we can then write m uniquely as $m = f(m') + n$ where $m' \in M'$ and $n \in N$. Define $\theta : M \to M' \oplus M''$ by $\theta(f(m') + n) = (m', gn)$. We claim that θ is an isomorphism. First we check that it is a homomorphism:

$$\begin{aligned}\theta(rm_1 + m_2) &= \theta(rf(m_1') + rn_1 + f(m_2') + n_2) = \theta(f(rm_1' + m_2') + (rn_1 + n_2)) \\ &= (rm_1' + m_2', g(rn_1 + n_2)) = r(m_1', gn_1) + (m_2', gn_2) \\ &= r\theta((m_1') + n_1) + \theta(f(m_2') + n_2)\end{aligned}$$

Now we show that θ has trivial kernel. If $m = f(m') + n \in \ker\theta$, then

$$\theta(m) = 0 \implies (m', gn) = 0 \implies m' = 0 \text{ and } gn = 0$$

Since N has trivial intersection with $\operatorname{im} f = \ker g$, this implies $n = 0$. Hence $m = f(m') + n = 0$, so θ has trivial kernel. Now we show θ is surjective. Let $(m', m'') \in M' \oplus M''$. Then there exists $m \in M$ so that $gm = m''$. Define $n = m - f(m')$. Then

$$\theta(f(m') + n) = (m', gn) = (m', g(m - f(m'))) = (m', gm) = (m', m'')$$

Thus θ is an isomorphism. Finally, we check that the required diagram (from case 3 of the above proposition) commutes.

$$\begin{aligned}\theta \circ f(m') &= \theta(f(m') + 0) = (m', 0) = \iota \circ \mathrm{id}_{M'} \\ \pi \circ \theta(m) &= \pi(\circ\theta(f(m') + n)) = \pi(m', gn) = gn = g(f(m') + n) = gm\end{aligned}$$

Thus by the equivalence already proved, there exists $\phi : M'' \to M$ so that $g \circ \phi = \mathrm{id}_{M''}$ (so the sequence splits).

Now we show that if such a submodule N exists, it is isomorphic to M''. We claim that the map $\alpha : N \to M''$ given by $n \mapsto gn$ is an isomorphism. It is a homomorphism since g is a homomorphism. The kernel of α is the intersection of N with the kernel of g, which is trivial since N intersects $\operatorname{im} f$ trivially. If $m'' \in M''$, then there is $m = f(m') + n \in M$ so that $m'' = gm = g(f(m') + n) = gn$. Thus α is surjective, so it is an isomorphism. $\square$

Proposition 0.6 (Exercise 4a, part 3)**.** *Let R be a ring and let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules that splits. By previous results, we have

$$M' \xleftarrow{\psi} M \xleftarrow{\phi} M''$$

so that $\psi f = \mathrm{id}_{M'}$ and $g\phi = \mathrm{id}_{M''}$. Then we have

$$M = \operatorname{im} f \oplus \ker\psi \qquad M = \ker g \oplus \operatorname{im}\psi \qquad M \cong M' \oplus M''$$

Proof. We already showed that if the sequence splits then $M \cong M' \oplus M''$. Lang shows on page 132 that $M = \ker g \oplus \operatorname{im} \phi$. We essentially repeat that argument to show $M = \operatorname{im} f \oplus \ker \psi$. If $m \in M$, then

$$\psi(m - f\psi m) = \psi m - \psi f \psi m = 0$$

so $m - f\psi m \in \ker \psi$. Thus $m = f\psi m + (m - f\psi m)$ so $M = \operatorname{im} f + \ker \psi$. We need to show that the sum is direct. If $m = f(m') + p \in \operatorname{im} f + \ker \psi$, then applying ψ gives

$$\psi m = \psi(f(m') + p) = \psi f(m') = m'$$

Thus $f(m')$ is uniquely determined by m, so by injectivity of f, m' is uniqely determined by m. Thus the sum is direct. □

Proposition 0.7 (Exercise 4b, part one)**.** *Let E and E_i for $i \in \{1, \ldots, m\}$ be modules over a ring R. Let $\phi_i : E_i \to E$ and $\psi_i : E \to E_i$ be homomorphisms such that*

$$\psi_i \phi_i = \mathrm{id}_{E_i} \qquad \psi_i \phi_j = 0 \text{ when } i \neq j \qquad \sum_{i=1}^{m} \phi_i \psi_i = \mathrm{id}_E$$

Then the maps $\psi : E \to \prod_i E_i$ given by $x \mapsto (\psi_i x)$ and $\phi : \prod_i E_i \to E$ given by $(x_i) \mapsto \sum_i \phi_i x_i$ are both isomorphisms. (In fact, they are inverse to each other.)

Proof. If we can show that both ψ and ϕ are R-module homomorphisms that are inverse to each other, then they must be isomorphisms. First we show that ψ and ϕ are R-module homomorphisms. Let $x, y \in E$ and $(x_i), (y_i) \in \prod_i E_i$ and $r \in R$.

$$\psi(rx + y) = (\psi_i(rx + y)) = (r\psi_i x + \psi_i y) = r(\psi_i x) + (\psi_i y) = r\psi x + \psi y$$
$$\phi(r(x_i) + (y_i)) = \phi((rx_i + y_i)) = \sum_i \phi_i(rx_i + y_i) = r\sum_i \phi_i x_i + \sum_i \phi_i y_i = r\phi(x_i) + \phi(y_i)$$

(Keep in mind that all sums are over finitely many indices.) Now we show that $\phi\psi = \mathrm{id}_E$.

$$\phi\psi(x) = \phi(\psi_i x) = \sum_i \phi_i \psi_i x = x$$

And finally we show that $\psi\phi = \mathrm{id}_{\prod_i E_i}$.

$$\psi\phi(x_i) = \psi\left(\sum_i \phi_i x_i\right) = \sum_i \psi\phi_i x_i = \sum_i (\psi_j \phi_i x_i)$$

Since $\psi_j \phi_i = 0$ for $i \neq j$, the $j \neq i$ terms are all zero. So the only nonzero term is $(\psi_i \phi_i x_i) = (x_i)$. (Recall that $\psi_i \phi_i = \mathrm{id}_{E_i}$.) Thus ψ and ϕ are both invertible R-module homomorphisms with R-linear inverses, so they are isomorphisms. □

Proposition 0.8 (Exercise 4b, part two)**.** *Let E be an R-module such that E is a direct product of finitely many R-modules E_i. Let $\phi_i : E_i \to E$ be the inclusion $x_i \mapsto (0, \ldots, x_i, \ldots, 0)$ and $\psi_i : E \to E_i$ be the projection $x \mapsto x_i$, then*

$$\psi_i \phi_i = \mathrm{id}_{E_i} \qquad \psi_i \phi_j = 0 \text{ when } i \neq j \qquad \sum_{i=1}^{m} \phi_i \psi_i = \mathrm{id}_E$$

Proof. These are pretty obvious. For the sake of completeness:

$$\begin{aligned}
\psi_i\phi_i(x_i) &= \psi_i(0,\ldots,x_i,\ldots,0) = x_i \\
\psi_i\phi_j(x_j) &= \psi_i(0,\ldots,x_j,\ldots,0) = 0 \\
\left(\sum_i \phi_i\psi_i\right)(x) &= \sum_i \phi_i\psi_i(x) = \sum_i \phi_i(x_i) = \sum_i (0,\ldots,x_i,\ldots,0) = x
\end{aligned}$$

□

Proposition 0.9 (Exercise 9a)**.** *Let A be a commutative ring and let M be an A-module. Let S be a multiplicative subset of A. As with localization of rings, we define*

$$S^{-1}M = \left\{\frac{m}{s} : m \in M, s \in S\right\}$$

with an equivalence relation $\frac{m}{s} \sim \frac{m'}{s'}$ when there exists $t \in S$ so that $t(ms' - m's) = 0$. Define addition and multiplication in $S^{-1}M$ by

$$\begin{aligned}
\frac{m}{s} + \frac{m'}{s'} &= \frac{ms' + m's}{ss'} \\
\left(\frac{m}{s}\right)\left(\frac{m'}{s'}\right) &= \frac{mm'}{ss'}
\end{aligned}$$

Define an action $A \times S^{-1}M \to S^{-1}M$ by

$$a \cdot \left(\frac{m}{s}\right) = \frac{a \cdot m}{s}$$

Then $S^{-1}A$ is an A-module.

Proof. Let $a, a' \in A, m, m' \in M$, and $s, s' \in s$. The module action distributes over addition:

$$\begin{aligned}
a\left(\frac{m}{s} + \frac{m'}{s'}\right) &= a\frac{ms' + m's}{ss'} = \frac{a(ms' + m's)}{ss'} \\
&= \frac{ams' + am's}{ss'} = \frac{ams'}{ss'} + \frac{am's}{ss'} = a\frac{m}{s} + a\frac{m'}{s'}
\end{aligned}$$

And the addition in A distributes over the action:

$$(a + a') \cdot \frac{m}{s} = \frac{(a + a')m}{s} = \frac{am + a'm}{s} = \frac{am}{s} + \frac{a'm}{s} = a\frac{m}{s} + a'\frac{m}{s}$$

The action is associative:

$$a \cdot \left(a' \cdot \frac{m}{s}\right) = a \cdot \frac{a'm}{s} = \frac{aa'm}{s}$$

And the unit 1_A acts as identity:

$$1_A \cdot \frac{m}{s} = \frac{1_A m}{s} = \frac{m}{s}$$

□

Lemma 0.10 (for Exercise 9b)**.** *Let A be a commutative ring and M and A-module. Let S be a multiplicative subset of A, and define $S^{-1}M$ as above. If $f : M' \to M$ is an A-module homomorphism, we define the induced map $\widetilde{f} : S^{-1}M' \to S^{-1}M$ by $\frac{m'}{s} \mapsto \frac{f(m')}{s}$. Then $\widetilde{f}$ is an A-module homomorphism.*

Proof. Let $m', n' \in M'$ and $s, t \in S$. We check that $\widetilde{f}$ preserves addition.

$$\begin{aligned}\widetilde{f}\left(\frac{m'}{s} + \frac{n'}{t}\right) &= \widetilde{f}\left(\frac{m't + sn'}{st}\right) = \frac{f(m't + sn')}{st} = \frac{tf(m') + sf(n')}{st} \\ &= \frac{tf(m')}{st} + \frac{sf(n')}{st} = \frac{f(m')}{s} + \frac{f(n')}{t} = \widetilde{f}\left(\frac{m'}{s}\right) + \widetilde{f}\left(\frac{n'}{t}\right)\end{aligned}$$

And we check that $\widetilde{f}$ is compatible with the action from A.

$$\widetilde{f}\left(a \cdot \frac{m'}{s}\right) = \widetilde{f}\left(\frac{am'}{s}\right) = \frac{f(am')}{s} = \frac{af(m')}{s} = a \cdot \frac{f(m')}{s} = a \cdot \widetilde{f}\left(\frac{m'}{s}\right)$$

Thus $\widetilde{f}$ is an A-module homomorphism. $\square$

Proposition 0.11 (Exercise 9b)**.** *Let A be a commutative ring and M and A-module. Let S be a multiplicative subset of A, and define $S^{-1}M$ as above. If we have an exact sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Then the induced sequence

$$0 \longrightarrow S^{-1}M' \xrightarrow{\widetilde{f}} S^{-1}M \xrightarrow{\widetilde{g}} S^{-1}M'' \longrightarrow 0$$

is exact.

Proof. To show that the induced sequence is exact, we need to show $\ker \widetilde{f} = \{0\}, \operatorname{im} \widetilde{f} = \ker \widetilde{g}$, and $\operatorname{im} \widetilde{g} = S^{-1}M''$. First we consider $\ker \widetilde{f}$.

$$\ker \widetilde{f} = \left\{\frac{m'}{s} \in S^{-1}M' : \widetilde{f}\left(\frac{m'}{s}\right) = 0\right\} = \left\{\frac{m'}{s} \in S^{-1}M' : \frac{f(m')}{s} = 0\right\}$$

Suppose $\frac{m'}{s} \in \ker \widetilde{f}$. Then

$$\widetilde{f}\left(\frac{m'}{s}\right) = \frac{0}{s} \iff \frac{f(m')}{s} = \frac{0}{s} \iff \exists t \in S, stf(m') = f(stm') = 0$$

By exactness of the first sequence, $\ker f = 0$, so $stm' = 0$. Thus

$$\exists t \in S, stm' = 0 \iff \frac{m'}{s} = \frac{0}{s}$$

Thus $\ker \widetilde{f} = 0$. Now we consider $\operatorname{im} \widetilde{f}$.

$$\operatorname{im} \widetilde{f} = \left\{\widetilde{f}\left(\frac{m'}{s}\right) : \frac{m'}{s} \in S^{-1}M'\right\} = \left\{\frac{f(m')}{s} : m' \in M', s \in S\right\}$$

By exactness of the first sequence, $\operatorname{im} f = \ker g$, so we have

$$\begin{aligned}\operatorname{im} \widetilde{f} &= \left\{ \frac{m}{s} : m \in \ker g, s \in S \right\} = \left\{ \frac{m}{s} \in S^{-1}M : g(m) = 0 \right\} \\ &= \left\{ \frac{m}{s} \in S^{-1}M : g(m) = 0 \right\} = \left\{ \frac{m}{s} \in S^{-1}M : \widetilde{g}\left(\frac{m}{s}\right) = 0 \right\} = \ker \widetilde{g}\end{aligned}$$

Finally, we show that $\operatorname{im} \widetilde{g} = S^{-1}M''$. By exactness of the first sequence, $\operatorname{im} g = M''$, so

$$\begin{aligned}\operatorname{im} \widetilde{g} &= \left\{ \widetilde{g}\left(\frac{m}{s}\right) : m \in M, s \in S \right\} = \left\{ \frac{g(m)}{s} : m \in M, s \in S \right\} \\ &= \left\{ \frac{m''}{s} : m'' \in M'', s \in S \right\} = S^{-1}M''\end{aligned}$$

$\square$